COHOMOLOGICAL INTERSECTION THEORY AND DERIVED ALGEBRAIC GEOMETRY

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Abstract. The goal of these lecture notes is to explain a cohomological approach to intersection theory on schemes and stacks. By working with Voevodsky's theory of motivic cohomology, we see how this recovers and extends the cycle-theoretic approach of Fulton (and its extension to stacks by Kresch). Moreover, we use the language of derived algebraic geometry to systematically incorporate non-transverse phenomena, most notably Kontsevich's virtual fundamental classes. We discuss examples related to curve counting, cohomological Hall algebras, and even Shimura varieties. No prior knowledge of stacks or derived geometry is assumed.

Introduction

Let X be an oriented smooth manifold of dimension d. When X is not compact, we may use Borel–Moore homology [BM] to formulate Poincaré duality as follows:

Theorem 0.1. *There exists a fundamental class*

$$[X] \in \mathrm{H}_d^{\mathrm{BM}}(X, \mathbf{Z})$$

such that the induced homomorphism

$$[X] \cap (-) : \mathrm{H}^{d-*}(X, \mathbf{Z}) \to \mathrm{H}_*^{\mathrm{BM}}(X, \mathbf{Z})$$

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is invertible. Here $\cap$ denotes the cap product, the natural action of cohomology on Borel–Moore homology.

Given an oriented smooth submanifold $Y \subseteq X$ of dimension m, we also have a fundamental class

$$[Y] \in \mathrm{H}^{\mathrm{BM}}_m(X, \mathbf{Z})$$

by direct image along the inclusion $Y \hookrightarrow X$.

Recall that cohomology is also equipped with a cup product

$$\cup : \mathrm{H}^p(X, \mathbf{Z}) \otimes \mathrm{H}^q(X, \mathbf{Z}) \to \mathrm{H}^{p+q}(X, \mathbf{Z}).$$

This is Poincaré dual to an *intersection product*

$$\cdot : \mathrm{H}^{\mathrm{BM}}_m(X, \mathbf{Z}) \otimes \mathrm{H}^{\mathrm{BM}}_n(X, \mathbf{Z}) \to \mathrm{H}^{\mathrm{BM}}_{m+n-d}(X, \mathbf{Z}).$$

This name comes from the following "cohomological Bézout theorem":

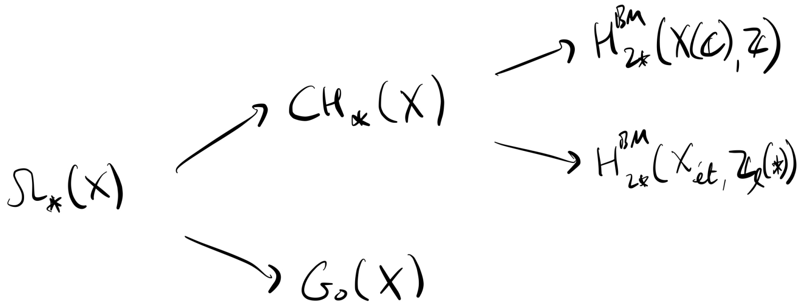
Theorem 0.2. *Let Y and Z be oriented submanifolds of X of dimension m and n, respectively. If Y and Z intersect* transversally*, then $Y \cap Z$ is an oriented submanifold of dimension $m+n-d$, and we have*

$$[Y] \cdot [Z] = [Y \cap Z] \tag{0.3}$$

in $\mathrm{H}^{\mathrm{BM}}_{m+n-d}(X, \mathbf{Z})$.

Moreover, Thom's transversality theorem [Tho] guarantees that in the non-transverse case, we may perturb $Y \rightsquigarrow Y'$ such that $[Y] = [Y']$ and Y' does intersect Z transversally. However, this approach will not be robust enough for our purposes, because there is no analogue of Thom transversality in many other contexts, such as equivariant geometry (where there is a finite group acting on X, say) or in algebraic geometry (where X is an algebraic variety).

In algebraic geometry, there are many Borel–Moore type theories.



Namely, for a quasi-projective algebraic variety X over a field k:

(a) *Topological Borel–Moore.* If k is a field with a complex embedding $k \hookrightarrow \mathbf{C}$, then we may consider the Borel–Moore homology of the space of complex points $X(\mathbf{C})$ (with the analytic topology).

(b) *ℓ-adic Borel–Moore.* If k is a field and ℓ is a prime different from the characteristic of k, then we may consider the ℓ-adic Borel–Moore homology. This is defined following Grothendieck as cohomology with coefficients in the ℓ-adic dualizing complex [SGA5, Lau].

(c) *Chow or motivic Borel–Moore.* We may consider the Chow homology $\mathrm{CH}_*(X)$, whose elements are algebraic cycles (linear combinations of integral subvarieties) up to rational equivalence. This is equipped with cycle class maps to topological and étale Borel–Moore homology. Moreover, it can also be identified with Voevodsky's theory of motivic Borel–Moore homology (see [MWV, Prop. 19.18][1]):

$$\mathrm{CH}_*(X) \simeq \mathrm{H}^{\mathrm{BM}}_{2*}(X_{\mathrm{mot}}, \mathbf{Z}(*)).$$

(d) *G-theory.* We may consider the G-theory $\mathrm{G}_0(X)$, i.e., the algebraic K-theory of coherent sheaves on X [SGA6]. (This is sometimes also called K-theory, but is typically different from $\mathrm{K}_0(X)$, the K-theory of vector bundles, when X is not smooth; the latter behaves like *cohomology* rather than Borel–Moore homology.) Using motivic homotopy theory, this can also be realized as a generalized motivic Borel–Moore homology theory (see [Jin]). With rational coefficients, there is a Grothendieck–Riemann–Roch transformation inducing an isomorphism from G-theory to Chow homology ([BFM, Chap. III, §1]).

(e) *Bordism.* We may consider the algebraic bordism $\Omega_*(X)$. This is defined using motivic homotopy theory as the generalized motivic Borel–Moore homology theory defined by Voevodsky's cobordism spectrum (see [Voe, Lev2]). When k is of characteristic zero, it can be alternatively defined via generators and relations as in [LM]. This interpolates between Chow homology and G-theory in some sense (see [Hoy] and [SØ, Thm. 1.2]).

In algebraic geometry, the analogue of Theorem 0.2 also holds in each of the above theories. However, since there is no analogue of Thom transversality, (0.3) does not uniquely characterize the intersection product in this setting. For example, in Chow homology, Chow's moving lemma implies that the intersection product is uniquely characterized by the more general formula

$$[Y] \cdot [Z] = \sum_\alpha m_\alpha [W_\alpha]$$

for all integral subvarieties Y and Z intersecting *properly* (without excess), where W_α are the irreducible components of the intersection $Y \cap Z$, and m_α are the intersection multiplicities (see e.g. [Ful]). The latter are equal to 1 in the transverse case, but are very subtle to define in general; the correct definition was eventually obtained after work of Severi, Weil, Chevalley, Samuel, and Serre (see e.g. [FM]).

[1] If k is of characteristic $p > 0$, then this is only known after tensoring both sides with $\mathbf{Z}[1/p]$; see [CD, Cor. 8.12].

An alternative approach to non-transverse intersections comes from derived algebraic geometry, via the following result (see [Kha1]):

Theorem 0.4. *Let X be a smooth algebraic variety over a field k. Let Y and Z be smooth subvarieties of X. Then we have*

$$[Y]\cdot[Z] = [Y \overset{\mathbf{R}}{\underset{X}{\times}} Z] \tag{0.5}$$

in any of the above Borel–Moore–type homology theories, where $Y \times^{\mathbf{R}}_X Z$ is the derived intersection*, and $[Y \times^{\mathbf{R}}_X Z]$ is a virtual fundamental class in the sense of* [Kon].

In the case of complex algebraic varieties and topological Borel–Moore homology, this result was announced without proof by J. Lurie in [Lur, Chap. 0].

The formula (0.5) remains valid when Y and Z are smooth varieties that are just projective over X. In Chow theory, this generalized formula uniquely determines the intersection product (at least up to inverting p in the case where k is of characteristic $p > 0$). This follows by a simple resolution of singularities argument, or by using alterations in the characteristic p case.

In these notes, we will see a construction of the virtual fundamental class appearing on the right-hand side, as well as a proof of this formula. For simplicity, we will restrict our attention to the topological, ℓ-adic, and Chow homology theories. Motivated by moduli theory, we will also work in the generality of algebraic stacks.

1. Cohomology of stacks

1.1. **Sheaves.** We will work with any of the following sheaf theories:

Let k be a field. Consider one of the following sheaf theories on schemes of finite type over k:

(a) *Topological.* Suppose k is of characteristic zero with a complex embedding $k \subseteq \mathbf{C}$ and Λ is a commutative ring. Define $\mathbf{D}(X)$ to be the derived ∞-category $\mathrm{D}_{\mathrm{top}}(X, \Lambda)$ of sheaves of Λ-modules on the topological space $X(\mathbf{C})$.

(b) *Étale.* Let Λ be a ring of positive characteristic n, e.g. $\mathbf{Z}/n\mathbf{Z}$, where n is prime to the characteristic of k. Define $\mathbf{D}(X)$ to be the derived ∞-category $\mathrm{D}_{\mathrm{\acute{e}t}}(X, \Lambda)$ of sheaves of Λ-modules on the small étale site of X. (Alternatively, we could take the ℓ-adic derived ∞-category.)

(c) *Motivic.* Let Λ be a commutative ring, and assume either that k is of characteristic zero or that its characteristic is invertible in Λ. Define $\mathbf{D}(X)$ to be the ∞-category $\mathrm{DM}(X, \Lambda)$ of Λ-linear Voevodsky motives over X, defined as in [CD]. We denote the unit object by $\Lambda_X \in \mathbf{D}(X)$ (it represents Voevodsky's motivic cohomology).

Each of these sheaf theories is equipped with the six operations. Thus, for any morphism $f : X \to Y$ there is an adjoint pair

$$f^* : \mathbf{D}(Y) \to \mathbf{D}(X), \quad f_* : \mathbf{D}(X) \to \mathbf{D}(Y).$$

If f is of finite type, we also have the compactly supported variants

$$f_! : \mathbf{D}(X) \to \mathbf{D}(Y), \quad f^! : \mathbf{D}(Y) \to \mathbf{D}(X).$$

Finally, we have the bifunctors $\otimes$ (tensor product) and $\underline{\mathrm{Hom}}$ (internal Hom) on $\mathbf{D}(X)$ for every X. These operations are subject to several compatibilities. For example, the base change formula gives for any cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

a canonical isomorphism

$$q^* f_! \simeq f'_! p^*$$

of functors $\mathbf{D}(X) \to \mathbf{D}(Y')$. Similarly, we have the projection formula

$$f_!(\mathcal{F}) \otimes \mathcal{G} \simeq f_!(\mathcal{F} \otimes f^*(\mathcal{G}))$$

for any finite type morphism f and any $\mathcal{F} \in \mathbf{D}(X)$, $\mathcal{G} \in \mathbf{D}(Y)$. There is a "forget supports" map

$$f_! \to f_*$$

which is invertible when f is proper.

It will be convenient to introduce the following notation: for an integer $n \in \mathbf{Z}$, we will write

$$\mathcal{F}\langle n\rangle := \mathcal{F}(n)[2n]$$

for any object $\mathcal{F} \in \mathbf{D}(X)$. Here (n) denotes the Tate twist (which can be ignored in the topological case) and $[2n]$ the usual shift.

1.2. **Cohomology and Borel–Moore homology.**

Notation 1.1. For any object $\mathcal{F} \in \mathbf{D}(X)$, we write

$$R\Gamma(X, \mathcal{F}) \in \mathbf{D}(\Lambda)$$

for the derived global sections of $\mathcal{F}$. Here $\mathbf{D}(\Lambda)$ denotes the ∞-category of complexes of Λ-modules. We have

$$\mathrm{H}^i\, R\Gamma(X, \mathcal{F}) = \mathrm{H}^i(X, \mathcal{F}) = \mathrm{Hom}_{\mathbf{D}(X)}(\Lambda_X, \mathcal{F}[i]).$$

Definition 1.2. Let X be a scheme of finite type over k. We write $a : X \to \mathrm{Spec}(k)$ for the structural morphism.

(i) The *cohomology* of X is the complex

$$\mathrm{C}^\bullet(X) := R\Gamma(X, \Lambda).$$

(ii) The *Borel–Moore homology* of X is

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X) \coloneqq R\Gamma(X, a^!(\Lambda)).$$

(Recall that $a^!(\Lambda)$ is a dualizing complex for X.) Its cohomology groups are denoted

$$\mathrm{H}^{\mathrm{BM}}_n(X,\Lambda) \coloneqq \mathrm{H}^{-n}\left(\mathrm{C}^{\mathrm{BM}}_\bullet(X)\right) = \mathrm{H}^{-n}(X, a^!(\Lambda))$$

for $n \in \mathbf{Z}$.

We also have relative variants. Given a finite type morphism $f : X \to Y$, we define the *relative* Borel–Moore homology of X over Y by

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y}) \coloneqq R\Gamma(X, f^!(\Lambda_Y)),$$

so that $\mathrm{C}^{\mathrm{BM}}_\bullet(X) = \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/k})$.

Example 1.3. In the motivic case, there is a canonical isomorphism

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -i\rangle \simeq z^{d-i}(X)_\Lambda$$

in $\mathbf{D}(\Lambda)$, for any d-dimensional scheme X of finite type over k, where the right-hand side is the Λ-linearized Bloch cycle complex. In particular,

$$\mathrm{H}^{\mathrm{BM}}_s(X)\langle -i\rangle \simeq \mathrm{CH}^{d-i}(X,s)_\Lambda$$

on homology groups, where the right-hand side is the Λ-linearized Bloch higher Chow group. Indeed, by Zariski descent (by Theorem 1.5 below for the left-hand side, and by localization for the Bloch cycle complex [Lev1] for the right-hand side), both sides are right Kan extended from affines. In particular, we may assume that X is quasi-projective, in which case the result is equivalent to [CD, Cor. 8.12].

1.3. **Functoriality.** Borel–Moore homology has proper push-forwards: given a proper morphism $f : X \to Y$, the counit $f_*f^! \simeq f_!f^! \to \mathrm{id}$ induces a direct image map

$$f_* : \mathrm{C}^{\mathrm{BM}}_\bullet(X) \to \mathrm{C}^{\mathrm{BM}}_\bullet(Y).$$

If f is smooth of relative dimension d then there is an isomorphism $f^*\langle d\rangle \simeq f^!$, whose right transpose $\mathrm{id} \to f_*f^!\langle -d\rangle$ induces a Gysin map

$$f^! : \mathrm{C}^{\mathrm{BM}}_\bullet(Y) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -d\rangle$$

in Borel–Moore homology.

Let $1 \in \mathrm{C}^\bullet(\mathrm{Spec}(k)) \simeq \mathrm{C}^{\mathrm{BM}}_\bullet(\mathrm{Spec}(k))$ be the unit. If X is smooth of relative dimension d, then its *fundamental class* is defined as[2]

$$[X] \coloneqq f^!(1) \in \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -d\rangle.$$

This induces the Poincaré duality isomorphism

$$(-) \cap [X] : \mathrm{C}^\bullet(X) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -d\rangle.$$

[2]Note the abuse of notation: if $K \in \mathbf{D}(\Lambda)$ is a complex, $x \in X$ means that x is a morphism $\Lambda \to K$ in $\mathbf{D}(\Lambda)$. Of course, x gives rise to an actual element of $\mathrm{H}^0(K)$.

Similarly, if $f : X \to Y$ is a smooth morphism of relative dimension d over a base S, then there is a Gysin map

$$f^! : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/S})\langle -d\rangle,$$

a relative fundamental class (take $Y = S$)

$$[X_{/Y}] := f^!(1) \in \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d\rangle,$$

and a relative Poincaré duality isomorphism

$$(-) \cap [X_{/Y}] : \mathrm{C}^\bullet(X) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d\rangle.$$

1.4. **Properties.** Here are some properties of these theories.

Theorem 1.4 (Localization)**.** *If $i : Z \hookrightarrow X$ is a closed immersion with complementary open immersion $j : U = X \smallsetminus Z \to X$, then there is an exact (distinguished) triangle*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(Z) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(X) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(U).$$

This gives rise to the usual long exact localization sequence. In the motivic case, the latter extends the right-exact sequence in Chow homology

$$\mathrm{CH}_*(Z) \xrightarrow{i_*} \mathrm{CH}_*(X) \xrightarrow{j^!} \mathrm{CH}_*(U) \to 0$$

to the left.

Theorem 1.5 (Descent)**.**

(i) *Let $p : Y \twoheadrightarrow X$ be a smooth surjection of relative dimension d. In the motivic case, assume either that $\Lambda \supseteq \mathbf{Q}$ or that p is a Zariski or Nisnevich cover. Then there is a homotopy limit diagram*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X) \xrightarrow{p^!} \mathrm{C}^{\mathrm{BM}}_\bullet(Y)\langle -d\rangle \rightrightarrows \mathrm{C}^{\mathrm{BM}}_\bullet(Y \underset{X}{\times} Y)\langle -2d\rangle \overset{\rightarrow}{\rightrightarrows} \cdots$$

in $\mathbf{D}(\Lambda)$.

(ii) *For any scheme X, there is a canonical isomorphism*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(X) \simeq \varprojlim_{(S,s)} \mathrm{C}^{\mathrm{BM}}_\bullet(S)$$

where the homotopy limit is taken over the category $\mathrm{Lis}^{\mathrm{aff}}_X$ of pairs (S, s) where S is an affine scheme and $s : S \to X$ is a smooth morphism.

Theorem 1.6 (Codescent)**.** *Let $p : Y \twoheadrightarrow X$ be a proper surjection. In the motivic case, assume either that $\Lambda \supseteq \mathbf{Q}$ or that p is a cdh cover. Then there is a homotopy colimit diagram*

$$\cdots \overset{\rightarrow}{\rightrightarrows} \mathrm{C}^{\mathrm{BM}}_\bullet(Y \underset{X}{\times} Y) \rightrightarrows \mathrm{C}^{\mathrm{BM}}_\bullet(Y) \xrightarrow{p_*} \mathrm{C}^{\mathrm{BM}}_\bullet(X)$$

in $\mathbf{D}(\Lambda)$.

In the motivic case, this extends the Kimura sequence in Chow homology [Kim, Thm. 1.8, Rmk. 1.9]

$$\mathrm{CH}_*(Y \underset{X}{\times} Y) \rightrightarrows \mathrm{CH}_*(Y) \xrightarrow{p_*} \mathrm{CH}_*(X) \to 0$$

to the left.

1.5. **Stacks.** A *stack* is a sheaf of groupoids on the category of affine schemes, i.e., a functor

$$\mathcal{X} : (\mathrm{Aff})^{\mathrm{op}} \to \mathrm{Grpd}.$$

Note that we only want to regard groupoids up to *equivalence* (rather than isomorphism), so Grpd here denotes the ∞-categorical localization of groupoids with respect to equivalences. (This ∞-category is 2-truncated, i.e., can safely be thought of as a 2-category.) The sheaf condition here means that for any family of smooth morphisms $(T_\alpha \twoheadrightarrow S)_\alpha$ with $T = \coprod_\alpha T_\alpha \to S$ surjective, the diagram

$$\mathcal{X}(S) \to \mathcal{X}(T) \rightrightarrows \mathcal{X}(T \underset{S}{\times} T) \overset{\to}{\underset{\to}{\to}} \mathcal{X}(T \underset{S}{\times} T \underset{S}{\times} T)$$

is a homotopy limit diagram. If we regard $\mathcal{X}$ as a moduli functor, the moral meaning of this is that the objects $\mathcal{X}$ classifies satisfy descent.

We say that $\mathcal{X}$ is an *Artin* stack if its diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by algebraic spaces) and there exists a smooth surjection $p : \coprod_\alpha X_\alpha \to \mathcal{X}$ where X_α are affine schemes. It is *Deligne–Mumford* if the same condition holds with p *étale*.

Given a point $x : \mathrm{Spec}(k) \to \mathcal{X}$, where k is a field, the *stabilizer* of $\mathcal{X}$ at x is the sheaf of groups

$$\mathrm{St}_x(\mathcal{X}) = \underline{\mathrm{Aut}}_{\mathcal{X}(\mathrm{Spec}(k))}(x)$$

of automorphisms of x in the groupoid $\mathcal{X}(\mathrm{Spec}(k))$. When $\mathcal{X}$ has representable diagonal, its stabilizers $\mathrm{St}_x(\mathcal{X})$ are representable by algebraic spaces, i.e., they are group algebraic spaces. When $\mathcal{X}$ is Artin, they are smooth. Moreover, an Artin stack $\mathcal{X}$ is Deligne–Mumford (resp. an algebraic space) if and only if its stabilizers $\mathrm{St}_x(\mathcal{X})$ are all finite étale (resp. trivial) groups. Recall that algebraic spaces are mild generalizations of schemes.

Other examples of Artin stacks include various moduli problems: moduli of curves, vector bundles, principal G-bundles for an algebraic group G, coherent sheaves, Higgs bundles, quiver representations, etc.

Typically, an Artin (resp. Deligne–Mumford) stack $\mathcal{X}$ can be written locally (in the Zariski or Nisnevich topology) as a quotient stack $[X/G]$, where G is a smooth (resp. finite étale) group scheme acting on a scheme X. Here $[X/G]$ is the moduli stack of principal G-bundles over X.

1.6. **Cohomology of stacks.** Let $\mathcal{X}$ be an Artin stack of finite type over k. We define the ∞-category $\mathbf{D}(\mathcal{X})$ by the homotopy limit

$$\mathbf{D}(\mathcal{X}) = \varprojlim_{(S,s)} \mathbf{D}(S) \tag{1.7}$$

over the category $\mathrm{Lis}_{\mathcal{X}}$ of pairs (S,s) where S is a scheme and $s : S \to X$ is a smooth morphism. Equivalently, we can use the full subcategory $\mathrm{Lis}_{\mathcal{X}}^{\mathrm{aff}}$ of such pairs where S is affine. Roughly speaking, an object $\mathcal{F} \in \mathbf{D}(\mathcal{X})$ is thus an object $\mathcal{F}_S \in \mathbf{D}(S)$ for every $(S,s) \in \mathrm{Lis}_{\mathcal{X}}$, an isomorphism $\mathcal{F}_S \simeq u^*\mathcal{F}_T$ for every morphism $u : (S,s) \to (T,t)$ in $\mathrm{Lis}_{\mathcal{X}}$ (so that $u \circ t = s$), and finally a homotopy coherent system of compatibilities between these isomorphisms. This construction is called the *lisse extension* (see [KR, §12]). It is straightforward to extend the six operations to Artin stacks with this definition (see [Kha1, App. A]). In the topological and étale cases, this construction agrees with previous constructions of Kapranov–Vasserot [KV] and Liu–Zheng [LZ], respectively.

We can now define cohomology and Borel–Moore homology of Artin stacks as in Subsect. 1.2. Equivalently, we have

$$\mathrm{C}^\bullet(\mathcal{X}) \simeq \varprojlim_{(S,s)} \mathrm{C}^\bullet(S)$$

and

$$\mathrm{C}_\bullet^{\mathrm{BM}}(\mathcal{X}) \simeq \varprojlim_{(S,s)} \mathrm{C}_\bullet^{\mathrm{BM}}(S)\langle -d_s\rangle$$

where d_s is the relative dimension of $s : S \to \mathcal{X}$. The functorialities and properties we stated for schemes now extend to Artin stacks.

For quotient stacks, this construction recovers equivariant cohomology:

Theorem 1.8. *Let X be a scheme of finite type over k with an action of an algebraic group G, and write $\mathcal{X} = [X/G]$ for the quotient stack. Then the cohomology and Borel–Moore homology of $\mathcal{X}$ can be computed by the Totaro–Morel–Voevodsky approximations to the Borel construction. In particular, in the motivic case we have*

$$\mathrm{H}_{2i}^{\mathrm{BM}}([X/G],\Lambda)(-i) = \mathrm{H}^0\,\mathrm{C}_\bullet^{\mathrm{BM}}([X/G])\langle -i\rangle \simeq \mathrm{CH}_i^G(X)_\Lambda.$$

Here on the right-hand side are the G-equivariant Chow groups of Edidin–Graham [EG] *(tensored with Λ).*

Kresch [Kre] has extended Chow homology to Artin stacks. Under a mild technical hypothesis, this also agrees with our construction:

Theorem 1.9. *Let $\mathcal{X}$ be an Artin stack of finite type over k with affine stabilizers. In the motivic case, there are isomorphisms*

$$\mathrm{H}_{2i}^{\mathrm{BM}}(\mathcal{X},\Lambda)(-i) = \mathrm{H}^0\,\mathrm{C}_\bullet^{\mathrm{BM}}(\mathcal{X})\langle -i\rangle \simeq \mathrm{CH}_i(\mathcal{X}).$$

A proof of this comparison will appear in [BP]. For quotient stacks, both sides are isomorphic to the Edidin–Graham Chow groups. In general, $\mathcal{X}$ admits a stratification by quotient stacks and the claim follows by using the localization sequence. (In fact, the last step is much more involved because one needs to compare the boundary map in Borel–Moore homology with the explicit one defined by Kresch, which is highly nontrivial.)

2. Derived geometry and intersection theory

2.1. **Nonabelian derived categories.** In the usual machinery of homological algebra, which works well in (nice enough) abelian categories, one uses chain complexes to resolve objects. Dold and Puppe [DP] observed that, even in some nonabelian situations, one can still use *simplicial objects* to resolve, and that this still gives a well-behaved theory of derived functors. This led Quillen [Qui] to the theory of *homotopical algebra* (or "nonabelian homological algebra"), where the nonabelian derived category is defined by regarding simplicial objects up to weak homotopy equivalence. By the Dold–Kan equivalence, simplicial objects are equivalent to connective[3] chain complexes in the abelian case, so in that case the nonabelian derived category coincides with the usual one.

The machinery of nonabelian derived categories takes so-called *algebraic* categories (in the sense of [ARV]) as input. Roughly speaking, $\mathcal{A}$ is algebraic if there are enough "(finite) projective" objects. The technical meaning of this is that there is a full subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ of finite projective objects, such that $\mathcal{A}$ is freely generated by $\mathcal{A}_0$ under filtered colimits and reflexive coequalizers (a mild generalization of quotients by equivalence relations).

The nonabelian derived category of $\mathcal{A}$ can be described using the language of ∞-categories (see [HTT, §5.5.8]) as the ∞-category freely generated by $\mathcal{A}_0$ under filtered colimits and *geometric realizations* (i.e., homotopy colimits of simplicial diagrams).

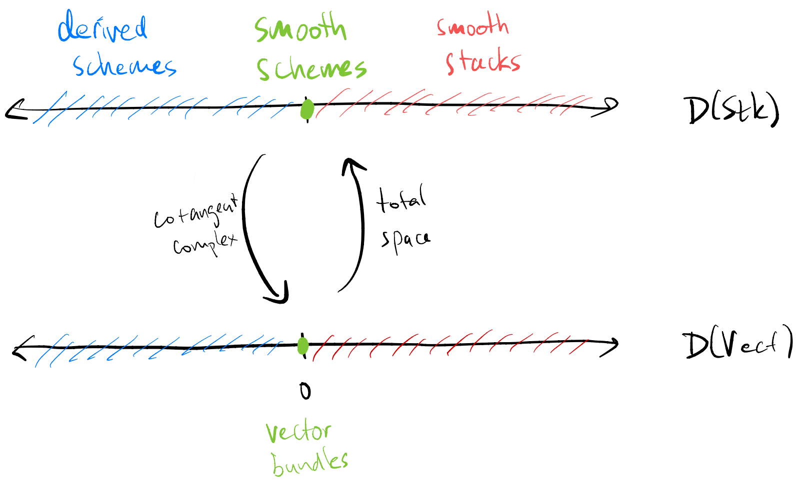
For example, the derived category of sets is the ∞-category of homotopy types (= "anima", following [ÇS]) or, equivalently, ∞-groupoids. If X is a set with an action of a group G, then the quotient X/G in sets can now be "animated" to an *animum* or ∞-groupoid $X/\!\!/G$ which is defined as the homotopy colimit of the diagram

$$\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G \times G \times X \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times X \rightrightarrows X \to X/\!\!/G.$$

As an ∞-groupoid, this can be described as follows: the objects of $X/\!\!/G$ are elements $x \in X$; isomorphisms $x \to y$ are elements $g \in G$ such that $g \cdot x = y$; and there are no nontrivial higher isomorphisms (in other words, this is a 1-groupoid). Note that we have $\pi_0(X/\!\!/G) = X/G$.

2.2. **Derived stacks.** As the name suggests, the category of derived stacks should be regarded, at least morally speaking, as the derived category of stacks. The "good" or "finite projective" objects are the smooth schemes. The picture is as follows.

[3] Meaning: homology concentrated in degrees $\geqslant 0$, or cohomology concentrated in degrees $\leqslant 0$.



We can form the "total space" of a complex of vector bundles as a derived stack; it will be underived (but stacky) if and only if the complex is coconnective[4], and it will be a scheme (but derived) if and only if the complex is connective. Conversely, we can "linearize" a derived stack by taking its cotangent complex (which is the nonabelian left-derived functor of the cotangent sheaf).

In fact, the nonabelian derived category construction cannot be applied to the category of stacks directly. However, the category of commutative rings is algebraic, with finite projectives given by the polynomial rings $\mathbf{Z}[t_1, \dots, t_n]$ ($n \geqslant 0$). Objects of its nonabelian derived ∞-category are called *derived commutative rings*. Its opposite is by definition the ∞-category of *affine derived schemes*. A *derived stack* is then an étale sheaf of ∞-groupoids on affine derived schemes, and we define *Artin* derived stacks by requiring existence of suitable atlases; see [Toë, §5.2] for details.

2.3. **The normal deformation.** Given a derived Artin stack X, the *conormal complex* is defined as the (-1)-shifted cotangent complex:

$$\mathcal{N}_X \coloneqq \mathcal{L}_X[-1].$$

In the world of derived stacks, we can form its "derived total space", which we call the *normal bundle* of X; this is denoted

$$N_X \coloneqq \mathbf{V}_X(\mathcal{N}_X)$$

and parametrizes *co*sections $\mathcal{N}_X \to \mathcal{O}_X$ (following Grothendieck's convention). Similarly, if $f : X \to Y$ is a morphism, then we write

$$\mathcal{N}_{X/Y} \coloneqq \mathcal{L}_{X/Y}[-1], \quad N_{X/Y} \coloneqq \mathbf{V}_X(\mathcal{N}_{X/Y}).$$

If f is a closed immersion between smooth schemes, then this is the usual normal bundle.

Recall that deformation to the normal cone associates, to any closed immersion $i : Z \to X$, a family of closed immersions over $\mathbf{A}^1$ which deforms i to the zero

[4]Meaning: homology concentrated in degrees $\leqslant 0$, or cohomology concentrated in degrees $\geqslant 0$.

section of the normal cone. Here is the derived version (constructed jointly with D. Rydh):

Theorem 2.1. *Let $f : X \to Y$ be a morphism of derived Artin stacks. Then there exists a commutative diagram of derived Artin stacks*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[d, "0"] & X \times \mathbf{A}^1 \arrow[d, "\widehat{f}"] & X \times \mathbf{G}_m \arrow[l] \arrow[d, "f\times\mathrm{id}"] \\ N_{X/Y} \arrow[r, "i"] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbf{G}_m \arrow[l, "j"'] \arrow[d, equal] \\ Y \arrow[r, "0"] & Y \times \mathbf{A}^1 & Y \times \mathbf{G}_m \arrow[l] \end{tikzcd} \tag{2.2}$$

where each square is homotopy cartesian.

The idea is to define $D_{X/Y}$ as the *derived Weil restriction* of $X \to Y$ along the inclusion $0 : Y \hookrightarrow Y \times \mathbf{A}^1$. This is a derived stack over $Y \times \mathbf{A}^1$ with the universal property that morphisms $S \to D_{X/Y}$ over $Y \times \mathbf{A}^1$ are in bijection with morphisms $S \times^{\mathbf{R}}_{\mathbf{A}^1} \{0\} \to X$ over Y. It is easy to see that this derived stack satisfies the desired properties, and the nontrivial part is the algebraicity (i.e., that it is Artin). See [Kha1, Thm. 1.3] in the quasi-smooth case, and [HP, Thm. 5.1.1, Prop. 5.1.14] and [HKR] for algebraicity results that apply to the general case.

Note that a variant of this construction in *formal* algebraic geometry appears in Simpson's work on nonabelian Hodge theory and in [GR, Vol. II, Chap. 9, §2]. It can be recovered by taking the formal completion, in the sense of [GR, Vol. II, Chap. 2, 3.1.3(iii)], of $D_{X/Y}$ along the morphism $\widehat{f}$.

2.4. **Specialization to the normal bundle.** Let $f : X \to Y$ be a morphism of derived Artin stacks, over a base derived Artin stack S. Consider the complementary pair of closed-open immersions

$$N_{X/Y} \xrightarrow{i} D_{X/Y} \xleftarrow{j} Y \times \mathbf{G}_m.$$

In the localization triangle

$$\mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S}) \xrightarrow{i_*} \mathrm{C}^{\mathrm{BM}}_\bullet(D_{X/Y}{}_{/S}) \xrightarrow{j^!} \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S}),$$

the boundary map

$$\partial : \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S})[-1] \to \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S})$$

gives rise to the *specialization map*

$$\begin{aligned} \mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) &\xrightarrow{\mathrm{incl}} \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S}) \oplus \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/S})(1)[1] \\ &\simeq \mathrm{C}^{\mathrm{BM}}_\bullet(Y \times \mathbf{G}_{m/S})[-1] \xrightarrow{\partial} \mathrm{C}^{\mathrm{BM}}_\bullet(N_{X/Y}{}_{/S}), \end{aligned} \tag{2.3}$$

where the splitting comes from the unit section $1 : Y \to Y \times \mathbf{G}_m$.

Variant 2.4. Let X be a classical Artin stack. The normal complex $\mathcal{N}_X$ is almost never perfect, unless X is smooth or at least lci. Instead, there is a specialization to the normal *cone*

$$\mathrm{C}_\bullet^{\mathrm{BM}}(X) \to \mathrm{C}_\bullet^{\mathrm{BM}}(\mathfrak{C}_X)$$

defined using an underived version of the deformation $D_{X/Y}$ (see [Ma, Thm. 2.31] and [AP, Thm. 7.2]). Here $\mathfrak{C}_X$ is the underived version of N_X, the so-called *intrinsic normal cone* of [BF, AP].

Suppose we are given a closed immersion $e : \mathfrak{C}_X \hookrightarrow E$ where E is a derived vector bundle on X (i.e., the derived total space of a perfect complex). Then there is a specialization map

$$\mathrm{C}_\bullet^{\mathrm{BM}}(X) \to \mathrm{C}_\bullet^{\mathrm{BM}}(\mathfrak{C}_X) \xrightarrow{e_*} \mathrm{C}_\bullet^{\mathrm{BM}}(E). \tag{2.5}$$

For example, if there is a derived structure on X, i.e., a derived Artin stack $\widetilde{X}$ whose classical truncation $\widetilde{X}_{\mathrm{cl}}$ is identified with X, then we have such an immersion $e : \mathfrak{C}_X \to N_{\widetilde{X}}|_X$ into the normal bundle of $\widetilde{X}$ (restricted to X). The specialization map (2.5) in this case coincides with the derived one constructed above.

2.5. **Virtual fundamental classes.**

Definition 2.6. Let $f : X \to Y$ be a morphism of derived Artin stacks. We say that f is *quasi-smooth* if $X_{\mathrm{cl}} \to Y_{\mathrm{cl}}$ is locally of finite presentation, and the relative cotangent complex $\mathcal{L}_{X/Y}$ is perfect of Tor-amplitude $\leqslant 1$ (homologically).

In the quasi-smooth case, the normal complex $\mathcal{N}_{X/Y}$ is of Tor-amplitude $\leqslant 0$, and hence the normal bundle $N_{X/Y}$ is smooth (a so-called *vector bundle stack*). We have the following generalized Thom isomorphism:

Proposition 2.7. *Let E be a vector bundle stack on X, i.e., $E = \mathbf{V}_X(\mathcal{E})$ for a perfect complex $\mathcal{E}$ of Tor-amplitude $\leqslant 0$. Then the Gysin map for the projection $\pi : E \to X$*

$$\pi^! : \mathrm{C}_\bullet^{\mathrm{BM}}(X_{/S})\langle r\rangle \to \mathrm{C}_\bullet^{\mathrm{BM}}(E_{/S})$$

is an isomorphism, where r is the virtual rank of $\mathcal{E}$.

Thus for a *quasi-smooth* morphism f, say of relative virtual dimension d, we have a canonical isomorphism

$$\mathrm{C}_\bullet^{\mathrm{BM}}(N_{X/Y\,/S}) \simeq \mathrm{C}_\bullet^{\mathrm{BM}}(X_{/S})\langle -d\rangle.$$

Combining this with specialization produces now the *Gysin map*

$$f^! : \mathrm{C}_\bullet^{\mathrm{BM}}(Y_{/S}) \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}_\bullet^{\mathrm{BM}}(N_{X/Y\,/S}) \simeq \mathrm{C}_\bullet^{\mathrm{BM}}(X_{/S})\langle -d\rangle.$$

If f is *smooth*, then this is the same as the Gysin map from the first lecture.

Taking $Y = S$, the image of the unit $1 \in \mathrm{C}^\bullet(Y) \simeq \mathrm{C}_\bullet^{\mathrm{BM}}(Y_{/Y})$ gives rise to the relative *virtual fundamental class*

$$[X_{/Y}] \in \mathrm{C}_\bullet^{\mathrm{BM}}(X_{/Y})\langle -d\rangle. \tag{2.8}$$

Again, in the smooth case, this is the same as the relative fundamental class from the first lecture.

Remark 2.9. Unlike the smooth case, the virtual fundamental class does not satisfy Poincaré duality in general. However, for *regular* schemes, regarded as quasi-smooth derived schemes, this does hold: see [Kha2].

Let us try to understand the constructions N_X and D_X a little better in the quasi-smooth case. First of all, suppose that X is *smooth*. Let $T_X = \mathbf{V}_X(\mathcal{L}_X)$ be the tangent bundle. (When X is a scheme or Deligne–Mumford stack, $\mathcal{L}_X$ is the sheaf Ω^1_X of algebraic Kähler differentials; but beware that if X is 1-Artin, then T_X is 2-Artin.) Then we have

$$N_X = T_X[1] = [X/T_X].$$

Here the quotient stack $[X/T_X]$ is formed with respect to the additive group structure on T_X, which we regard as acting trivially on X. The notation $T_X[1]$ just means $\mathbf{V}_X(\mathcal{L}_X[-1])$ (recall that under the Grothendieck convention, $\mathbf{V}_X(-)$ is contravariant).

Now consider the quasi-smooth case. We assume that X is a scheme for simplicity. The quasi-smooth condition implies that, Zariski-locally on X, there exists a smooth scheme M, a vector bundle $E \to M$, a section $s : M \to E$, and a homotopy cartesian square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & M \arrow[d, "s"] \\ M \arrow[r, "0"] & E. \end{tikzcd}$$

That is, X is the derived intersection of s with the zero section. When this intersection is not *transverse*, this is different from the usual scheme-theoretic intersection, and there is a nontrivial derived structure on X. In this local model, $\mathcal{L}_X$ is the two-term complex

$$\mathcal{L}_X = \left[\mathcal{E} \xrightarrow{d} \Omega_M\right]$$

where $E = \mathbf{V}_M(\mathcal{E})$. Thus the normal bundle N_X is the quotient

$$N_X = [E/T_M].$$

Here are the basic properties of the virtual fundamental class.

Theorem 2.10. (i) Functoriality. *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-smooth morphisms of derived Artin stacks, of relative virtual dimensions d and e, respectively. Then we have*

$$[X_{/Y}] \circ [Y_{/Z}] \simeq [X_{/Z}]$$

in $\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Z})\langle -d-e\rangle$, where $\circ$ denotes the composition product

$$\circ : \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d\rangle \otimes \mathrm{C}^{\mathrm{BM}}_\bullet(Y_{/Z})\langle -e\rangle \to \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Z})\langle -d-e\rangle.$$

(ii) Base change. *Suppose given a homotopy cartesian square of derived Artin stacks*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where f is quasi-smooth of relative virtual dimension d. Then there is a canonical homotopy

$$q^*_\Delta[X_{/Y}] \simeq [X'_{/Y'}] \in \mathrm{C}^{\mathrm{BM}}_\bullet(X'_{/Y'})\langle -d\rangle$$

where q^*_Δ denotes the change of base map

$$q^*_\Delta : \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/Y})\langle -d\rangle \to \mathrm{C}^{\mathrm{BM}}_\bullet(X'_{/Y'})\langle -d\rangle.$$

(iii) Excess intersection. *Suppose given a commutative square of derived Artin stacks*

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"] \arrow[dr, phantom, "\Delta"] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{2.11}$$

where f and g are quasi-smooth of relative virtual dimension d and e. Assume that Δ is cartesian on classical truncations and that the fibre $\mathcal{E}$ (the excess sheaf*) of the canonical map*

$$p^*\mathcal{N}_{X/Y} \to \mathcal{N}_{X'/Y'}$$

is of Tor-amplitude $[0,0]$. Then there is a canonical homotopy

$$q^*_\Delta[X_{/Y}] \simeq e(\mathcal{E}) \cap [X'_{/Y'}] \in \mathrm{C}^{\mathrm{BM}}_\bullet(X'_{/Y'})\langle -d\rangle,$$

where $e(\mathcal{E})$ is the Euler class (= top Chern class) of $\mathcal{E}$.

2.6. **The non-transverse Bézout theorem.** Let $f : Z \to X$ be a quasi-smooth proper morphism of relative virtual dimension $-d$. The relative fundamental class $[Z_{/X}] \in \mathrm{C}^{\mathrm{BM}}_\bullet(Z_{/X})\langle d\rangle$ defines, by proper push-forward, a class in cohomology:

$$f_*[Z_{/X}] \in \mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X})\langle d\rangle \simeq \mathrm{C}^\bullet(X)\langle d\rangle,$$

which we denote simply by $[Z]$.

Theorem 2.12. *Let $f : Y \to X$ and $g : Z \to X$ be quasi-smooth proper morphisms of relative virtual dimension $-d$ and $-e$. Then there is a canonical homotopy*

$$[Y] \cup [Z] \simeq [Y \overset{\mathbf{R}}{\underset{X}{\times}} Z] \in \mathrm{C}^\bullet(X)\langle d+e\rangle.$$

Theorem 2.12 follows formally from the functoriality and base change properties of the virtual fundamental class (Theorem 2.10). Indeed, form the homotopy cartesian square

$$\begin{array}{ccc} Y \times^{\mathbf{R}}_X Z & \xrightarrow{q} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X. \end{array}$$

Let $h : Y \times^{\mathbf{R}}_X Z \to X$ denote the diagonal composite. We can compute $[Y \times^{\mathbf{R}}_X Z] = h_*[W_{/X}]$ using the functoriality axiom, in terms of $[W_{/Y}]$ and $[Y_{/X}]$. But by the base change axiom, $[W_{/Y}]$ is the base change of $[Z_{/X}]$. Using the properties of the composition product, one gets the formula

$$h_*[W_{/X}] \simeq f_*[Y_{/X}] \circ g_*[Z_{/X}]$$

in $\mathrm{C}^{\mathrm{BM}}_\bullet(X_{/X})\langle d\rangle$, which is equivalent to the asserted formula in $\mathrm{C}(X)\langle d\rangle$. See [Kha1, Thm. 3.22] for more details.

Example 2.13. Suppose X is a smooth scheme, so that we have a Chow ring $\mathrm{CH}^*(X)$. Let Y and Z be smooth (or lci) subvarieties of X. Then Theorem 2.12 gives (in the motivic case),

$$[Y] \cup [Z] = [Y \overset{\mathbf{R}}{\underset{X}{\times}} Z] \in \mathrm{CH}^*(X)_\Lambda$$

as asserted in the introduction. More generally, this holds for Y and Z schemes that are proper and smooth (or lci) over X. This generalized form of the formula uniquely determines the intersection product on $\mathrm{CH}^*(X)_\Lambda$, since by resolution of singularities (or alterations in the positive characteristic case), such classes generate the Chow group.

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