Motivic sheaves on algebraic stacks

Adeel Khan

2025-01-14

Academia Sinica

Motivic sheaves (Morel, Voevodsky, Ayoub, Cisinski–Déglise)

For schemes X, there are triangulated categories $\mathbf{SH}(X)$ of motivic sheaves.

These are equipped with the six operations: $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$, $f^!$.

Borel–Moore homology theories

Let $E \in \mathbf{SH}(k)$ be a motivic spectrum.

For $f : X \to \mathrm{Spec}(k)$, define

$$\mathrm{H}^{\mathrm{BM}}_s(X; E)(r) := \mathrm{Hom}_{\mathbf{SH}(k)}(\mathbf{1}(r)[s], f_* f^!(E)).$$

$E = H\mathbf{Z} \rightsquigarrow \mathrm{CH}_*(-, *)$: higher Chow (Voevodsky, Cisinski–Déglise)

$E = \mathrm{KGL} \rightsquigarrow \mathrm{G}_*(-)$: G-theory (Jin)

$E = \mathrm{MGL} \rightsquigarrow \Omega_*(-)$: algebraic bordism (Levine)

Intersection theory can be done in terms of the six operations: fundamental classes, Gysin maps, intersection products, Chern classes, GRR, ... (Déglise, Déglise-Jin-Kh.)

Equivariant and stacky intersection theory

For a linear algebraic group G, we have the following theories for schemes with G-action:

- $\mathrm{G}^G(-)$ (Thomason)
- $\mathrm{CH}_*^G(-,*)$ (Totaro, Edidin–Graham)*
- $\Omega_*^G(-,*)$ (Heller–Malagon-Lopez, Krishna)*

*on G-quasi-projective schemes

Let $\mathcal{X}$ be an Artin stack.

- $\mathrm{G}(\mathcal{X})$ Gillet, Toën
- $\mathrm{CH}_*(\mathcal{X})$ Kresch*

*on 1-Artin stacks with affine stabilizers

Program

1. Extend triangulated categories of motivic sheaves from schemes to (derived) stacks.
2. Prove theorems about motivic sheaves on derived stacks.
3. Read off constructions of equivariant and stacky intersection theories and results about them.

Outline

I. Motivic sheaves on (derived) stacks

II. Equivariant intersection theories

III. Intersection theories on stacks

IV. A derived/stacky Fourier duality

Motivic sheaves on (derived) stacks

Motivic sheaves on derived schemes

Theorem (Kh. 2016)

For any derived scheme X with classical truncation X_{cl}, there is a canonical equivalence $\mathbf{SH}(X) \simeq \mathbf{SH}(X_{\mathsf{cl}})$ which commutes with the six operations.

Theorem (Lurie 2004)

There is a canonical equivalence $\mathbf{D}_{\text{ét}}(X) \simeq \mathbf{D}_{\text{ét}}(X_{\mathsf{cl}})$ which commutes with the six operations.

Lisse extension

Let $\mathbf{D}$ be a sheaf theory, such as $\mathbf{SH}$, $\mathbf{DM}$, ..., $\mathbf{D}_{\text{ét}}$, $\mathbf{D}_{\text{Bet}}$, ...

Let $\mathcal{X}$ be a (derived) stack, and set

$$\mathrm{Lis}_{\mathcal{X}} := \{(S, s) : S \text{ is a scheme, } s : S \to \mathcal{X} \text{ a } \textit{smooth} \text{ morphism}\}.$$

Define (Hoyois–Kh. '20)

$$\mathbf{D}^{\triangleleft}(\mathcal{X}) := \varprojlim_{(S,s) \in \mathrm{Lis}_{\mathcal{X}}} \mathbf{D}(S)$$

where the limit is taken over $*$-pullbacks.

Informally, a sheaf $\mathcal{F}$ on $\mathcal{X}$ is a collection of sheaves $\mathcal{F}_S \in \mathbf{D}(S)$ for all $(S, s) \in \mathrm{Lis}_{\mathcal{X}}$, compatible up to coherent homotopy.

The six operations

Let $\mathbf{D}$ be a motivic sheaf theory satisfying étale descent, such as $\mathbf{SH}_{\text{ét}}$, $\mathbf{DM}_{\text{ét}}$, $\mathbf{DM}_{\mathbf{Q}}$, or any classical sheaf theory such as $\mathbf{D}_{\text{ét}}$, $\mathbf{D}_{\text{Bet}}$, ...

Theorem (Liu–Zheng 2012, Richarz–Scholbach 2019, Kh. 2019)

On Artin stacks, the extension $\mathbf{D}^{\triangleleft}(-)$ admits the six operations $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$, $f^!$. Moreover, $\mathbf{D}^{\triangleleft}(-)$ is the unique extension of $\mathbf{D}(-)$ satisfying Čech descent along smooth surjections.

Nis-Artin stacks

Recall that general motivic sheaf theories like **SH** only satisfy *Nisnevich* descent, i.e., Čech descent along étale morphisms that are surjective on all field-valued points.

By definition, an Artin stack $\mathcal{X}$ has a cover $p : X \twoheadrightarrow \mathcal{X}$ where X is a scheme and p is a smooth morphism admitting étale-local sections.

Say $\mathcal{X}$ is Nis-Artin if it p can be chosen to admit *Nisnevich*-local sections.

This turns out to be automatic for quasi-separated 1-Artin stacks with separated diagonal (Laumon–Moret-Bailly).

The six operations without étale descent

Let $\mathbf{D}$ be any motivic sheaf theory, such as $\mathbf{SH}$, $\mathbf{DM}$ $(:= \mathbf{D}_{H\mathbf{Z}})$, or $\mathbf{D}_{\mathrm{MGL}}$.

Theorem (Kh. '22)

On Nis-Artin stacks, the extension $\mathbf{D}^{\triangleleft}(-)$ admits the six operations $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$, $f^!$. Moreover, $\mathbf{D}^{\triangleleft}(-)$ is the unique extension of $\mathbf{D}(-)$ satisfying Čech descent along smooth morphisms admitting Nis-local sections.

For $\mathbf{D} = \mathbf{SH}$, Chowdhury '21 also studied $\mathbf{SH}^{\triangleleft}$ and constructed the !-operations for *representable* morphisms.

Genuine motivic homotopy theory for stacks

Let $\mathcal{X}$ be a *scalloped* stack, built Nisnevich-locally out of quotient stacks $[X/G]$ where G is linearly reductive and U is G-quasi-projective. For example, $\mathcal{X}$ is qcqs 1-Artin with affine diagonal over a field and has *linearly reductive* stabilizers.

There is another construction $\mathbf{SH}(\mathcal{X})$ (Kh.–Ravi '21) which specializes to the genuine-equivariant stable motivic homotopy category $\mathbf{SH}^G(X)$ (Hoyois '17).

Theorem (Kh.-Ravi '21)

On scalloped stacks, the categories $\mathbf{SH}(\mathcal{X})$ admit the operations $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_* for arbitrary morphisms, and $f_!$, $f^!$ for representable morphisms.

Equivariant intersection theory

Equivariant Borel–Moore homology

Let $E \in \mathbf{SH}(k)$ be a motivic spectrum.

Let X be a locally of finite type k-scheme and G an algebraic group acting on X.

For $f : [X/G] \to BG := [\mathrm{Spec}(k)/G]$ the induced morphism, define

$$\mathrm{H}_s^{\mathrm{BM},G}(X;E)(-r) := \mathrm{Hom}_{\mathbf{SH}(BG)}(\mathbf{1}(r)[s], f_* f^!(E_{BG}))$$

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X;E)\langle -v\rangle := \mathrm{Maps}_{\mathbf{SH}(BG)}(\mathbf{1}\langle v\rangle, f_* f^!(E_{BG}))$$

where $E_{BG} = E|^*_{BG}$ and $v \in \mathrm{K}_0(BG)$ is a virtual vector bundle.

Proposition

The Borel–Moore homology theory $\mathrm{C}_{\bullet}^{\mathrm{BM},G}(-;E)$ *admits equivariant proper push-forwards, equivariant smooth pull-backs, long-exact localization sequences, Euler classes of equivariant vector bundles, Chern classes when* E *is oriented, etc.*

Relation to the Borel construction

Let G be a linear algebraic group acting on X.

Let $(V_n)_n$ be a tower of G-representations. Let $W_n \subset V_n$ be G-invariant closed subschemes such that G acts freely on each $U_n := V_n \setminus W_n$, $U_n \subset U_{n+1}$, and $\mathrm{codim}_{V_n}(W_n)$ tends to ∞.

Theorem (Kh.–Ravi '22)

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X;E) \simeq \varprojlim_n \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_n; E)\langle -\Omega_{U_n} + \Omega_G \rangle$$

If E is oriented:

$$\mathrm{C}_\bullet^{\mathrm{BM},G}(X;E) \simeq \varprojlim_n \mathrm{C}_\bullet^{\mathrm{BM}}(X \overset{G}{\times} U_n; E)\langle -d_n + g \rangle$$

where $d_n = \dim(U_n/G)$ *and* $g = \dim(G)$.

Relation to the Borel construction, continued

Suppose E is $H\mathbf{Z}$ (or $H\mathbf{Z}[1/p]$ in char. p), $\mathrm{KGL}_{\mathbf{Q}}$, or $\mathrm{MGL}_{\mathbf{Q}}$.

Theorem (Kh.–Ravi '22)

For sufficiently large n,

$$\mathrm{H}^{\mathrm{BM},G}_{s}(X;E) \simeq \mathrm{H}^{\mathrm{BM}}_{s+2d_n-2g}(X \overset{G}{\times} U_n; E).$$

Comparison with classical constructions

Let Λ be a commutative ring in which $\mathrm{char}(k)$ is invertible.

Corollary (Kh.–Ravi '22)

$$\mathrm{H}^{\mathrm{BM},G}_{s+2n}(X; H\Lambda)(-n) \simeq \mathsf{CH}^G_n(X, s)_\Lambda.$$

Corollary (Kh.–Ravi '22)

$$\mathrm{H}^{\mathrm{BM},G}_{2n}(X; \mathrm{MGL})(-n) \twoheadrightarrow \Omega^G_n(X),$$
$$\mathrm{H}^{\mathrm{BM},G}_{2n}(X; \mathrm{MGL}_{\mathbf{Q}})(-n) \simeq \Omega^G_n(X)_{\mathbf{Q}},$$

Localization in equivariant bordism

It is not known whether $\Omega_*^G(-)$ admits right-exact localization sequences.

Theorem

For every G-invariant closed subscheme $Z \subseteq X$ with open complement U, there is an exact triangle

$$\cdots \xrightarrow{\partial} \mathrm{H}^{\mathrm{BM},G}_{2n+s}(Z;E)(-n) \xrightarrow{i_*} \mathrm{H}^{\mathrm{BM},G}_{2n+s}(X;E)(-n) \xrightarrow{j^*} \mathrm{H}^{\mathrm{BM},G}_{2n+s}(U;E)(-n) \xrightarrow{\partial} \cdots$$

Corollary (Kh.-Ravi '22)

There is a right-exact localization sequence

$$\Omega_n^G(Z)_{\mathbf{Q}} \xrightarrow{i_*} \Omega_n^G(X)_{\mathbf{Q}} \xrightarrow{j^*} \Omega_n^G(U)_{\mathbf{Q}} \to 0.$$

Lisse-extended equivariant G-theory

$$\mathrm{G}^{G,\triangleleft}(X) := \mathrm{G}^{\triangleleft}([X/G]) := \varprojlim_{(S,s)} \mathrm{G}(S) \qquad \text{over } (S,s) \in \mathrm{Lis}_{[X/G]}$$

Corollary (Kh.–Ravi '22)

$$\mathrm{C}_{\bullet}^{\mathrm{BM},G}(X;\mathrm{KGL}) \simeq \mathrm{G}^{G,\triangleleft}(X)$$

The map $\mathrm{G}([X/G]) \to \mathrm{G}^{\triangleleft}([X/G]) = \mathrm{G}^{G,\triangleleft}(X)$ is a *completion* at the augmentation ideal in $\mathrm{K}_0(BG) = R(G)$ (Krishna, Carlsson–Joshua).

Equivariant GRR

Theorem (Edidin–Graham '98, Krishna '14, Kh.–Ravi '22)

$$\mathsf{G}^{G,\triangleleft}(X)_{\mathbf{Q}} \simeq \prod_{n\in\mathbf{Z}} \mathrm{C}_{\bullet}^{\mathrm{BM},G}(X;H\mathbf{Q})\langle n\rangle.$$

$$\mathsf{G}_s^{G,\triangleleft}(X)_{\mathbf{Q}} \simeq \prod_{n\in\mathbf{Z}} \mathsf{CH}_n^G(X,s)_{\mathbf{Q}}.$$

Intersection theories on stacks

Borel–Moore homology on stacks

Let $E \in \mathbf{SH}(k)$.

Let $\mathcal{X}$ be an Artin stack and $f : \mathcal{X} \to \mathrm{Spec}(k)$.

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X}; E)\langle -v\rangle := \mathrm{Maps}_{\mathbf{SH}(k)}(\mathbf{1}\langle v\rangle, f_* f^!(E)),$$
$$\mathrm{H}^{\mathrm{BM}}_{s}(\mathcal{X}; E)(-r) := \mathrm{Hom}_{\mathbf{SH}(k)}(\mathbf{1}(r)[s], f_* f^!(E)).$$

These have proper* push-forwards, smooth pull-backs, long-exact localization sequences, Euler classes of vector bundles, Chern classes when E is oriented, etc.

Kresch Chow groups

Let $\mathcal{X}$ be a 1-Artin stack of finite type over k. There are cycle class maps

$$\mathrm{CH}_n(\mathcal{X})_\Lambda \to \mathrm{H}^{\mathrm{BM}}_{2n}(\mathcal{X}; H\Lambda)(-n).$$

Theorem (Kh., Bae–Park)

These are isomorphisms if (a) $\mathcal{X}$ is a global quotient, (b) $\mathcal{X}$ is DM and $\Lambda \supseteq \mathbf{Q}$, (c) $\mathcal{X}$ is smooth, or (d) the characteristic exponent of k is invertible in Λ.

Deformation to the normal cone in DAG Hekking–Kh.–Rydh

Let $f : X \to Y$ be a quasi-smooth morphism of derived stacks. There is a commutative diagram of cartesian squares

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "0"] & X \times \mathbf{A}^1 \arrow[d, "\widehat{f}"] & X \times \mathbf{G}_m \arrow[l] \arrow[d, "f \times \mathrm{id}"] \\
N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbf{G}_m \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m. \arrow[l]
\end{tikzcd}$$

Thus $\widehat{f}$ degenerates f to the zero section of the derived normal bundle $N_{X/Y} = T_{X/Y}[1]$.

Specialization map

For any quasi-smooth morphism of derived Artin stacks $f : X \to Y$, there is a canonical map

$$\mathrm{sp}_{X/Y} : \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y; E) \to \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y}; E)$$

defined using the localization triangle for

$$N_{X/Y} \hookrightarrow D_{X/Y} \hookleftarrow Y \times \mathbf{G}_m.$$

Virtual pullbacks

The normal bundle $N_{X/Y}$ is a vector bundle stack over X of rank $-\operatorname{vdim}(X/Y)$, so there is a generalized Thom isomorphism

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(X;E) \simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y};E)\langle d\rangle.$$

For any quasi-smooth morphism of derived Artin stacks $f : X \to Y$, we define

$$f^! : \mathrm{C}^{\mathrm{BM}}_{\bullet}(Y;E) \xrightarrow{\mathrm{sp}_{X/Y}} \mathrm{C}^{\mathrm{BM}}_{\bullet}(N_{X/Y};E) \simeq \mathrm{C}^{\mathrm{BM}}_{\bullet}(X;E)\langle -d\rangle$$

when E is oriented.

This generalizes Gysin pullbacks in intersection theory from the case of quasi-projective local complete intersection morphisms.

Note that, even if X and Y are schemes, this construction passes through the derived stacks $N_{X/Y}$ and $D_{X/Y}$ (which are not schemes unless $f : X \to Y$ is a closed immersion).

Similarly, if X and Y are 1-Artin, we need to make use of the extension of $\mathbf{D}(-)$ and the six operations to *higher* Artin stacks.

Virtual classes Kontsevich '95, Behrend–Fantechi '97, Kh. '19

Let X be a quasi-smooth derived Artin stack of virtual dim d.

For the projection $f : X \to \mathrm{Spec}(k)$ we get the virtual pull-back

$$f^! : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathrm{Spec}(k)) \to \mathrm{C}^{\mathrm{BM}}_\bullet(X)\langle -d\rangle$$

and hence to the canonical element

$$[X]^{\mathrm{vir}} \in \mathrm{C}^{\mathrm{BM}}_\bullet(X; E)\langle -d\rangle \quad \rightsquigarrow \quad [X]^{\mathrm{vir}} \in \mathrm{H}^{\mathrm{BM}}_{2d}(X; E)(-d)$$

called the *virtual fundamental class* of X.

- This general construction of virtual fundamental classes is useful in enumerative geometry (Pardon, Porta–Yu, ...) and "arithmetic" enumerative geometry (Feng–Yun–Zhang, Madapusi).
- The construction of virtual pull-backs is used in geometric representation theory (Kapranov–Vasserot, Mellit–Minets–Schiffmann–Vasserot, ...).
- The existence of suitable homology theories for higher Artin stacks themselves is also useful in enumerative geometry (Joyce, ...).

Intersection products

Virtual fundamental classes are actually useful in classical intersection theory, too.

If X is a smooth k-scheme, the cap product in cohomology gives rise by Poincaré duality to an intersection product

$$\mathrm{C}_\bullet^{\mathrm{BM}}(X)\langle -p\rangle \otimes \mathrm{C}_\bullet^{\mathrm{BM}}(X)\langle -q\rangle \to \mathrm{C}_\bullet^{\mathrm{BM}}(X;E)\langle -p-q+d\rangle.$$

If $f : Y \to X$ is proper quasi-smooth of virtual dim d, we have the VFC $f_*[Y]^{\mathrm{vir}}$ in $\mathrm{C}_\bullet^{\mathrm{BM}}(X;E)\langle -d\rangle$.

Theorem (Non-transverse Bézout formula)

Let Y and Z be smooth or lci closed subvarieties of X, of dimension p and q respectively. We have

$$[Y] \cdot [Z] \simeq [Y \overset{R}{\underset{X}{\times}} Z]^{\mathrm{vir}}$$

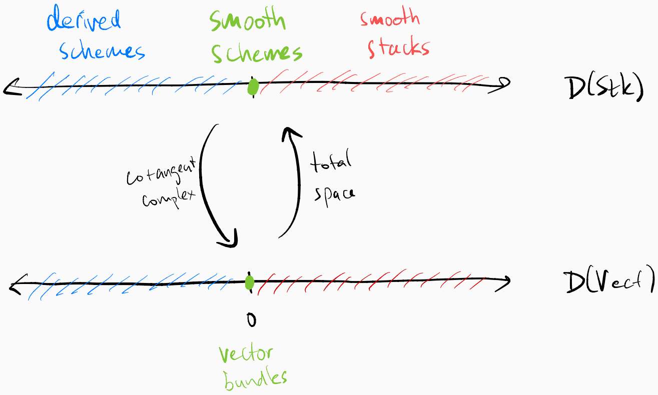
in $\mathrm{C}^{\mathrm{BM}}_{\bullet}(X; E)\langle -p-q+d \rangle$.

Even though the left-hand side consists of usual cycle classes, the right-hand side is genuinely virtual unless the intersection is transverse (that is to say, unless the derived intersection $Y \times^R_X Z$ reduces to the classical scheme-theoretic intersection).

Derived/stacky Fourier duality

Derived stacks

$\{\text{derived stacks}\} \approx$ derived category of $\{\text{smooth schemes}\}$



Sheaf-theoretic Fourier transform (Deligne, Laumon)

Let $E \to X$ be a vector bundle with dual $E^\vee \to X$.

$$\begin{tikzcd} & {[E^\vee/\mathbf{G}_m] \times [E/\mathbf{G}_m]} \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\ {[E^\vee/\mathbf{G}_m]} & & {[E/\mathbf{G}_m]} \end{tikzcd}$$

The Fourier transform for E is an equivalence

$$\mathsf{FT}_E : \mathbf{D}^{\mathbf{G}_m}(E) \to \mathbf{D}^{\mathbf{G}_m}(E^\vee)$$

$$\mathsf{FT}_E(-) := \mathrm{pr}_{2,!}(\mathrm{pr}_1^*(-) \otimes \mathcal{P}_E)$$

The kernel $\mathcal{P}_E$ is defined using the evaluation morphism $[E^\vee/\mathbf{G}_m] \times [E/\mathbf{G}_m] \to [\mathbf{A}^1/\mathbf{G}_m]$ and the sheaf $j_*(\mathbf{1}) \in \mathbf{D}^{\mathbf{G}_m}(\mathbf{A}^1)$.

Theorem (Laumon '03, Kh. '23)

The Fourier transform FT_E is involutive (up to a twist), and exchanges the $*$ and $!$ operations for any linear morphism of vector bundles.

Theorem (Kh. '23)

Let $E \to X$ be a derived vector bundle (perfect complex) and $E^\vee \to X$ the derived dual. Then FT_E, defined as before, satisfies the same properties as above.

Higher arithmetic theta series (Feng–Yun–Zhang '21)

Let X be a smooth projective curve over $\mathbf{F}_q$ and $X' \twoheadrightarrow X$ a finite étale double cover with automorphism $\sigma : X' \simeq X'$ over X.

Special cycles: $[Z^r_{\mathcal{E}}(a)]^{\mathrm{vir}} \in \mathrm{CH}_*(\mathrm{Sht}^r_{U(n)})_{\mathbf{Q}}$

Higher theta series: assemble these into a Fourier series

$$\Theta^r(\mathcal{G}, h, \mathcal{E}) \in \mathrm{CH}_*(\mathrm{Sht}^r_{U(n)})_{\mathbf{Q}}$$

for $\mathcal{G}$ a family of rank $2m$ vector bundles on X', $h : \mathcal{G} \to \sigma^*(\mathcal{G}^*)$ is a skew-Hermitian structure, and $\mathcal{E} \subseteq \mathcal{G}$ a Lagrangian subbundle.

Modularity

Conjecture

The higher theta series $\Theta^r(\mathcal{G}, h, \mathcal{E})$ are modular in the sense of automorphic forms. In other words, they are independent of the Lagrangian subbundle $\mathcal{E}$.

The case $r = 0$ amounts modularity of classical theta series.

Theorem (FYZ '22, Feng–Kh. '23)

Modularity holds on the generic fibre of $\mathrm{Sht}^r_{U(n)} \to (X')^r$.

Modularity of classical theta series is proven using Poisson summation, i.e., classical Fourier duality (Jacobi, Poisson). Our proof uses derived Fourier analysis, and a "sheaf-cycle correspondence" generalizing the classical sheaf-function correspondence.