Blow-ups and equivariant derived categories

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I. Derived categories of blow-ups

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d, "\text{blow-up}"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

Question:

How are invariants of $\tilde{X}$ related to invariants of X and Z?

Cohomology

Assume X and Z smooth, or $Z \hookrightarrow X$ lci.

$$H^n(\tilde{X}) \simeq H^n(X) \oplus \bigoplus_{i=1}^{c-1} H^{n-2i}(Z)$$

Categorification

$D^b_{coh}(X)$ = bounded derived category of coherent sheaves on X

$\cup\!|$

$\mathrm{Perf}(X)$ = perfect complexes on X

(locally $\simeq$ finite complexes of vector bundles on X)

Decategorification:

$$\mathrm{Perf}(X) \rightsquigarrow HP(\mathrm{Perf}(X)) \simeq \bigoplus_{i\in\mathbb{Z}} C^\bullet(X)[-2i]$$

$$D^b_{coh}(X) \rightsquigarrow HP(D^b_{coh}(X)) \simeq \bigoplus_{i\in\mathbb{Z}} C^{BM}_\bullet(X)[-2i]$$

$(X/\mathbb{C})$

HP = periodic cyclic homology

Semi-orthogonal decompositions (Thomason, Orlov):

Assume X and Z smooth or $Z \hookrightarrow X$ lci.

$$D^b_{coh}(\tilde{X}) = \langle D^b_{coh}(X), D^b_{coh}(Z), \dots, D^b_{coh}(Z) \rangle$$

$$\mathrm{Perf}(\tilde{X}) = \langle \mathrm{Perf}(X), \underbrace{\mathrm{Perf}(Z), \dots, \mathrm{Perf}(Z)}_{1, \dots, c-1} \rangle$$

Decategorify:

- Periodic cyclic homology: recover cohomology formula
- Algebraic K-theory: $K(\tilde{X}) \simeq K(X) \oplus \bigoplus_{i=1}^{c-1} K(Z)$

$$K_0(X) := \mathbb{Z}\{\mathcal{E}_\bullet \in \mathrm{Perf}(X)\} \Big/ \begin{matrix} [\mathcal{E}_\bullet] \sim [\mathcal{E}'_\bullet] + [\mathcal{E}''_\bullet] \\ \forall\ \mathcal{E}'_\bullet \to \mathcal{E}_\bullet \to \mathcal{E}''_\bullet \\ \text{exact triangle} \end{matrix}$$

Question:

What about non-smooth/lci centres?

Quasi-smoothness: (local picture)

$$\begin{tikzcd} Z \arrow[r, "\mathrm{lci}"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^n \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Z \arrow[r, "\mathrm{qsm}"] \arrow[d] \arrow[dr, phantom, "\boxed{R}"] & X \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & \mathbb{A}^n \end{tikzcd}$$

$Z = f^{-1}(0)$ f meeting $0 \in \mathbb{A}^n$ transversely

$\mathcal{O}_Z = \mathcal{O}_X/(f_1, \dots, f_n)$ $(f_1, \dots, f_n)$ regular sequence

$Z = Rf^{-1}(0)$ f arbitrary

$\mathcal{O}_Z = [\Lambda^n(\mathcal{O}_X^{\oplus n}) \to \cdots \to \Lambda^2(\mathcal{O}_X^{\oplus n}) \to \mathcal{O}_X^{\oplus n} \xrightarrow{f_1, \dots, f_n} \mathcal{O}_X]$

Derived blow-up: (local picture)

$$\begin{tikzcd} E = \mathbb{P}(N_{Z/X}) \arrow[r, hook] \arrow[d] & \tilde{X} \arrow[d] \\ Z \arrow[r, hook, "\text{qsm}"] & X \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\boxed{R}"] & \tilde{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{R}"] & X \arrow[d, "f"] \\ \mathbb{P}^{n-1} \arrow[r, hook] & \mathrm{Bl}_0 \mathbb{A}^n \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

Theorem (Kh. '18):

Assume $Z \hookrightarrow X$ quasi-smooth.

$$D^b_{coh}(\tilde{X}) = \langle D^b_{coh}(X), D^b_{coh}(Z), \dots, D^b_{coh}(Z) \rangle$$

$$\mathrm{Perf}(\tilde{X}) = \langle \mathrm{Perf}(X), \underbrace{\mathrm{Perf}(Z), \dots, \mathrm{Perf}(Z)}_{1, \dots, c-1} \rangle$$

Corollary:

$$H^n(\tilde{X}) \simeq H^n(X) \oplus \bigoplus_{i=1}^{c-1} H^{n-2i}(Z)$$

$$K(\tilde{X}) \simeq K(X) \oplus \bigoplus_{i=1}^{c-1} K(Z)$$

Corollary (descent by derived blow-ups):

H : noncommutative invariant (HH, HC, HP, K, ...)

$\tilde{X}$ = derived blow-up along quasi-smooth $Z \hookrightarrow X$

$$\begin{tikzcd} H(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & H(Z) \arrow[d] \\ H(\tilde{X}) \arrow[r] & H(E) \end{tikzcd}$$

"Mayer–Vietoris"

II. Equivariant blow-ups

Question:

How about the equivariant situation?

G : algebraic group

$G \circlearrowright X$ action preserving centre $Z \subseteq X$

$\rightsquigarrow$ induced action $G \circlearrowright \tilde{X}$

$H^*_G(X)$: G-equivariant cohomology

$D^{b,G}_{coh}(X)$: bounded derived category of G-equivariant sheaves

$\mathrm{Perf}^G(X)$: G-equivariant perfect complexes

Theorem (Krishna–Ravi '15, Kh. '18):

Assume $Z \hookrightarrow X$ quasi-smooth.

$$D^{b,G}_{coh}(\tilde{X}) = \langle D^{b,G}_{coh}(X),\ D^{b,G}_{coh}(Z), \dots, D^{b,G}_{coh}(Z) \rangle$$

$$\mathrm{Perf}^G(\tilde{X}) = \langle \mathrm{Perf}^G(X),\ \underbrace{\mathrm{Perf}^G(Z), \dots, \mathrm{Perf}^G(Z)}_{1, \dots, c-1} \rangle$$

Corollary:

$$HP^G(\tilde{X}) \simeq HP^G(X) \oplus \bigoplus_{i=1}^{c-1} HP^G(Z)$$

$$K^G(\tilde{X}) \simeq K^G(X) \oplus \bigoplus_{i=1}^{c-1} K^G(Z)$$

$$HP^G(X) := HP(\mathrm{Perf}^G(X))$$

⚠ $$HP^G(X) \neq C^\bullet_G(X)$$

"genuine" equivariance Borel-equivariant cochains

$$HP^G(X) \simeq \bigoplus_{g \in G} C^\bullet_G(X^g)$$

orbifold cohomology (Chen–Ruan)
if $G \circlearrowright X$ finite stabilizers
[Baranovsky '03 for X smooth]
[Kh.–Ravi, in progress]

Theorem (descent by equivariant derived blow-ups):

H : noncommutative invariant $(HH, HC, HP, K, \dots)$

$$\begin{tikzcd}
H^G(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & H^G(Z) \arrow[d] \\
H^G(\tilde{X}) \arrow[r] & H^G(E)
\end{tikzcd}$$

$G \curvearrowright \tilde{X}$ = derived blow-up along quasi-smooth $Z \hookrightarrow \overset{\overset{G}{\curvearrowright}}{X}$

Carefully bootstrapping from derived blow-ups, we can obtain a descent statement for usual blow-ups (along arbitrary centres)...

Pro-cdh descent (Morrow '16, Kerz-Strunk-Tamme '18):

$Z \subseteq X$ any closed subscheme

$\tilde{X}$ = classical blow-up

$$\begin{tikzcd}
K(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \varprojlim_n K(Z^{(n)}) \arrow[d] \\
K(\tilde{X}) \arrow[r] & \varprojlim_n K(E^{(n)})
\end{tikzcd}$$

Bachmann-Kh.-Ravi-Sosnilo '20:

- Holds for any noncommutative invariant (HH, HC, HP, K, K_{top}, ...)
- Holds G-equivariantly for G tame finite, split torus, ...
- For HP, K_{top}, no need to consider thickenings.

III. Applications of pro-cdh descent

Weibel conjecture: ('80)

Fact: $K_{-n}(X) = 0 \quad \forall n > 0$ if X smooth.

Weibel: For X a normal surface:

$\tilde{X}$: R.O.S.

$$\begin{cases} K_{-1}(X) = \operatorname{Pic}(E^{(n)})/\operatorname{Pic}(\tilde{X}) \\ K_{-2}(X) = \mathbb{Z}^{\lambda} & \lambda = \#\text{ "loops" in } E \\ K_{-n}(X) = 0 & \forall n > 2 \end{cases}$$

$$X = \operatorname{Spec}(k[x,y,z]/(x^3 + y^7 - z^2))$$

$K_{-1}(X) = k, \qquad K_{-n}(X) = 0 \qquad \forall n > 1$

Theorem (Kerz-Strunk-Tamme '17):

$$K_{-n}(X) = 0 \quad \forall n > \dim(X)$$

Theorem (Bachmann-Kh.-Ravi-Sosnilo '20):

$$K^G_{-n}(X) = 0 \quad \forall n > \dim(X) \qquad X \circlearrowleft G \text{ tame finite or split torus}$$

Proof: Pro-cdh descent + killing K-theory classes by blow-ups.

Noncommutative Hodge theory: (Kontsevich-Soibelman, Kaledin)

$\mathcal{C}$ dg-category / field k of char. 0

$HH(\mathcal{C})$: noncommutative Dolbeault cohomology

$HP(\mathcal{C})$: noncommutative de Rham cohomology

There exists a noncomm. Hodge-to-de Rham S.S. which degenerates when $\mathcal{C}$ is smooth proper/k.

Lattice conjecture: (Katzarkov - Kontsevich - Pantev '08, Toën, Blanc '12)

$$K^{top}(\mathcal{C}) \longrightarrow HP(\mathcal{C})$$ noncommutative Chern character

Conjecture:

$K^{top}(\mathcal{C}) \otimes \mathbb{C} \xrightarrow{\sim} HP(\mathcal{C})$ for $\mathcal{C}$ smooth proper

Conjecture:

Also holds for $\mathcal{C} = \mathrm{Perf}(\mathcal{X})$ for "reasonable" stacks $\mathcal{X}$.

Theorem (Kh. '23):

$K^{top}(\mathcal{C}) \otimes \mathbb{C} \xrightarrow{\sim} HP(\mathcal{C})$ holds for

- $\mathcal{C} = \mathrm{Perf}^G(X)$ G finite or torus
- $\mathcal{C} = \mathrm{Perf}(\mathcal{X})$ $\mathcal{X}$ DM stack (or Artin with diagonalizable stabilizers)

Proof: Cdh descent + equivariant resolution of singularities

$\rightsquigarrow$ reduce to smooth case [Halpern-Leistner–Pomerleano]